ABSTRACT

From P/E Ratio to Infinite Spreadsheet—
Mathematically Rigorous Derivations of the Zero-th and the First Order Solutions of the Rate of Return

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The P/E Ratio (Price/Earning) is one of the most popular concept in stock analysis, yet its exact interpretation is lacking. Most stock investors know the P/E Ratio as a financial indicator with the useful characteristics of being relatively time-invariant. In this paper, a rigorous mathematical derivation of the P/E Ratio is presented. The derivation shows that, in addition to its assumptions, the P/E Ratio can be considered the zero-th order solution to the rate of return on investment. The commonly used concept of the Capitalization Rate (Cap Rate = Net Income / Price) in real estate investment analysis can also be similarly derived as the zero-th order solution of the rate of return on real estate investment. This paper also derives the first order solution to the rate of return (Return = Dividend / Price + Growth) with its assumptions. Both the zero-th and the first order solutions are derived from the exact future accounting equation (Cash Return = Sum of Cash Flow + Cash From Resale). The exact equation has been used in the derivation of the exact solution of the rate of return. Empirically, as an illustration of an actual case, the rates of return are 3%, 73%, and 115% for a stock with 70% growth rate for, respectively, the zero-th order, the first order, and the exact solution to the rate of return; the stock doubled its price in 2004. This paper concludes that the zero-th, the first order, and the exact solution of the rate of return all can be derive mathematically from the same exact equation, which, thus, forms a rigorous mathematical foundation for investment analysis, and that the low order solutions have the very practical use in providing the analytically calculated initial conditions for the iterative numerical calculation for the exact solution.

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Introduction

Almost all the stock investors know about the P/E Ratio, and most stock analysts, including, particularly, Fed Chairman Alan Greenspan, use the equation return = dividend/price + growth ($r_0 = d_0E_0/P_0 + g_1$). In real estate, P/E ratio is equivalent to the inverse of the capitalization rate, Net Income/Price. Therefore, a rigorous mathematical derivation of these two equations is long overdue. Most importantly, the derivation starts from the same equation based on which the Infinite Spreadsheet is derived. The equation used to derive all three equations, namely, the zero-th order equation of P/E Ratio, the first order equation of $r_0 = d_0E_0/P_0 + g_1$, and the Infinite Spreadsheet is simply the exact realistic forward accounting of cash flow including the cash from resale.

P/E Ratio, instead of P/D Ratio, is used simply because D (Dividend) for many stocks is zero. The inadequacy of the zero-th order equation is obvious. P/E Ratio is unaffected by the growth rate of earning, in addition to the use of E which is the net income to the stock company, not of D which is the real cash return to the stock investor.

Now with all three equations available and mathematically interpreted, it might be interesting to compare their results. It should be noted that all three equations calculate the rate of return on investment to the stock investor. In this regard, Net Income is the term used in real estate and is, respectively, the earning and dividend for a company and a stock investor. Similarly, P/E or P/D ratio corresponds to the inverse of the Capitalization Rate in real estate, which is the Net Income divided by the price. The rate of return, especially for the stock market where the price in the form of the quote is given on the second, is really the only remaining unknown to be determined. Using the stock POT as an example, the zero-th equation P/E = 33, which translates into return = E/P = 3%, the first order return $r_0 = d_0E_0/P_0 + g_1$ = 73% and, the Infinite Spreadsheet calculates a return based on a 5-years growth rate of 70% to be around 115%. The Infinite Spreadsheet has, thus, calculated P/E Ratio and r_0 exactly. In 2004, POT doubled its price, a 100% increase.

The main reason that the three rates of return are all useful is that they are all approximately time-invariant, which can be used as market comparable inputs. However, only the Infinite Spreadsheet can calculate the absolute rate of return, with the other two being only relative indicators. Another practical advantage of the Infinite Spreadsheet is that it is completely mathematically rigorous and, thus, can take over the responsibility of analyst. On the other end, SEC can take the responsibility of the inputs to the Infinite Spreadsheet, leaving the analyst totally free of responsibility. In addition to being mathematical rigor, the Infinite Spreadsheet discloses all its methodology (in a patent) and equations, and all its inputs to infinity in time, resulting in full future accountability of the analysis.

Formal Classical Expansion:

$$f(x) = x^1 + x^2$$
$$x = \varepsilon^0 x_0 + \varepsilon^1 x_1 + \varepsilon^2 x_2$$

Substituting x into f(x),

$$f\,(x) = (\,x_0 + \varepsilon\,x_1 + \varepsilon^2\,x_2\,) + (\,x_0 + \varepsilon\,x_1 + \varepsilon^2\,x_2\,)^2$$

Usually, ε is a very small number, but, generally, ε can be any number. For example $\varepsilon = 0.01$, $\varepsilon^2 = 0.0001$, $\varepsilon^3 = 0.000001...$ In general, order, which is represented by the exponential of e, is related physically to the significance rather than the magnitude. For example, often the zero-th order terms are zero rather than finite, while the first order terms are finite.

$$\begin{array}{rrrr}
 & & & x_0 + \varepsilon\,x_1 + \varepsilon^2\,x_2 \\
 & & & x_0 + \varepsilon\,x_1 + \varepsilon^2\,x_2 \\
\hline
 & & \varepsilon^2\,x_0\,x_2 \;+ & \varepsilon^3\,x_1\,x_2 \;+\; \varepsilon^4\,x_2^2 \\
 & \varepsilon\,x_0\,x_1 \;+ & \varepsilon^2\,x_1^2 \;+ & \varepsilon^3\,x_1\,x_2 \\
x_0^2 \;+ & \varepsilon\,x_0\,x_1 \;+ & \varepsilon^2\,x_0\,x_2 & \\
\hline
\end{array}$$
$$\varepsilon^0\,x_0^2 + 2\,\varepsilon^1\,x_0\,x_1 + \varepsilon^2\,(\,2\,x_0\,x_2 + x_1^2\,) + 2\,\varepsilon^3\,x_1\,x_2 + \varepsilon^4\,x_2^2$$

Order	Epsilon (ε)	Collected Terms
0	ε^0	x_0^2
1	ε^1	$2\,x_0\,x_1$
2	ε^2	$2\,x_0\,x_2 + x_1^2$
3	ε^3	$2\,x_1\,x_2$
4	ε^4	x_2^2

$$\begin{aligned} f\,(x) &= (\,x_0 + \varepsilon\,x_1 + \varepsilon^2\,x_2\,) + (\,x_0 + \varepsilon\,x_1 + \varepsilon^2\,x_2\,)^2 \\ &= (\,x_0 + \varepsilon\,x_1 + \varepsilon^2\,x_2\,) + \varepsilon^0\,x_0^2 + 2\,\varepsilon^1\,x_0\,x_1 + \varepsilon^2\,(\,2\,x_0\,x_2 + x_1^2\,) + \\ &\quad 2\,\varepsilon^3\,x_1\,x_2 + \varepsilon^4\,x_2^2 \end{aligned}$$

$$f(x) = \varepsilon^0\,(x_0 + x_0^2\,) + \varepsilon^1\,(\,x_1 + 2\,x_0\,x_1\,) + \varepsilon^2\,(\,x_2 + 2\,x_0\,x_2 + x_1^2\,) + \varepsilon^3\,(\,2\,x_1\,x_2) + \varepsilon^4\,x_2^2$$

Collected terms for f (x)

Order	Epsilon (ε)	Collected Terms
0	ε^0	$x_0 + x_0^2$
1	ε^1	$x_1 + 2\,x_0\,x_1$
2	ε^2	$x_2 + 2\,x_0\,x_2 + x_1^2$
3	ε^3	$2\,x_1\,x_2$
4	ε^4	x_2^2

Exact Equation (Cash flow equation):

CashInvestmentReturn = SumOfCashFlow + CashFromResale

or write out in detail

$$(Price - Loan - BuyerExpense) * (1 + Return)^{InvestmentPeriod} = \Sigma\ (NetIncome - Loan * \%Payment - TaxBracket * NetIncome) + (ResalePrice - Loan - SellerExpense)$$

Write out in symbolic form:

$$(P - P * l - P * x) * (1 + r)^{T} = \Sigma\ (N - l * m - b * N) + P(Resale) - P * l - P(Resale) * e$$

where

- **P = Price or Resale Price**
- **l = loan as % of Price**
- **x = Expense of buyer as % of price**
- **r = Return on investment averaged over investment period T**
- **T = Investment period**
- **N = Net income**
- **m = Loan payment as % of Loan**
- **b = Tax bracket**
- **g = growth rate of price**
- **e = Expense of seller as % of price**

Our goal is to rigorously interpret the familiar concept of P/E Ratio and the commonly used equation for stock valuation

%Return = Dividend/Price + GrowthRate

in terms of their order of approximation. Practically, the resulting approximate equations can be used to calculate analytically the suitable initial conditions for the exact numerical calculation. Using classical expansion with the following substitutions for the zero-th and the first orders,

$$P = P_0 + P_1 = P_0 * (1 + g)^T = P_0 + P_0 * (g_1T + g_2T^2 + \ldots + g_nT^n + \ldots)$$
$$l = l_0 + l_1$$
$$x = x_0 + x_1$$
$$r = r_0 + r_1$$
$$N = N_0 + N_1$$
$$m = m_0 + m_1$$
$$b = b_0 + b_1$$
g = Defined implicitly in P above in a polynomial expansion
$$e = e_0 + e_1$$

where the zero-th order terms are constant and the first order terms can be time-varying, into the cash flow equation with 1 year investment period (T = 1)

$$[P_0 + P_1 - (P_0 + P_1)*(l_0 + l_1) - (P_0 + P_1)*(x_0 + x_1)] * (1+ r_0 + r_1) = N_0 + N_1 - (l_0 + l_1)*(m_0 + m_1) - (b_0 + b_1)*(N_0 + N_1) + P_0 + P_1(Resale) - (P_0 + P_1)*(l_0 + l_1) - (P_0 + P_1)*(e_0 + e_1)$$

Expanded out, we get

$P_0 + P_1 - P_0l_0 - P_0l_1 - P_1l_0 - P_1l_1 - P_0x_0 - P_0x_1 - P_1x_0 - P_1x_1 + P_0r_0 + P_1r_0 - P_0l_0r_0 - P_0l_1r_0 - P_1l_0r_0 - P_1l_1r_0 - P_0x_0r_0 - P_0x_1r_0 - P_1x_0r_0 - P_1x_1r_0 + P_0r_1 + P_1r_1 - P_0l_0r_1 - P_0l_1r_1 - P_1l_0r_1 - P_1l_1r_1 - P_0x_0r_1 - P_0x_1r_1 - P_1x_0r_1 - P_1x_1r_1 =$
$N_0 + N_1 - l_0m_0 - l_0m_1 - l_1m_0 - l_1m_1 - b_0N_0 - b_0N_1 - b_1N_0 - b_1N_1 + P_0 + P_1(\text{Resale}) - P_0l_0 - P_0l_1 - P_1l_0 - P_1l_1 - P_0e_0 - P_0e_1 - P_1e_0 - P_1e_1$

To construct stock P/E Ratio from the above expanded equation, we realize that for stock investment N = Dividend = DividendPayoutRatio * Earning = d * E = $N_0 + N_1 = d_0E_0 + d_0E_1 + d_1E_0 + d_1E_1$. Substituting Dividend for Net Income (N), we get

$P_0 + P_1 - P_0l_0 - P_0l_1 - P_1l_0 - P_1l_1 - P_0x_0 - P_0x_1 - P_1x_0 - P_1x_1 + P_0r_0 + P_1r_0 - P_0l_0r_0 - P_0l_1r_0 - P_1l_0r_0 - P_1l_1r_0 - P_0x_0r_0 - P_0x_1r_0 - P_1x_0r_0 - P_1x_1r_0 + P_0r_1 + P_1r_1 - P_0l_0r_1 - P_0l_1r_1 - P_1l_0r_1 - P_1l_1r_1 - P_0x_0r_1 - P_0x_1r_1 - P_1x_0r_1 - P_1x_1r_1 = d_0E_0 + d_0E_1 + d_1E_0 + d_1E_1 - l_0m_0 - l_0m_1 - l_1m_0 - l_1m_1 - b_0d_0E_0 - b_0d_0E_1 - b_0d_1E_0 - b_0d_1E_1 - b_1d_0E_0 - b_1d_0E_1 - b_1d_1E_0 - b_0d_1E_1 + P_0 + P_1(\text{Resale}) - P_0l_0 - P_0l_1 - P_1l_0 - P_1l_1 - P_0e_0 - P_0e_1 - P_1e_0 - P_1e_1$

To extract P/E or, more exactly, its zero-th order expression P_0/E_0 from the above equation, we need to set to zero all the non-zero order terms:

$P_0 - P_0l_0 - P_0x_0 + P_0r_0 - P_0l_0r_0 - P_0x_0r_0 = d_0E_0 - l_0m_0 - b_0d_0E_0 + P_0 - P_0l_0 - P_0e_0$

or dividing through by E_0, the above equation becomes

$P_0/E_0 - P_0l_0/E_0 - P_0x_0/E_0 + P_0r_0/E_0 - P_0l_0r_0/E_0 - P_0x_0r_0/E_0 = d_0 - l_0m_0/E_0 - b_0d_0 + P_0/E_0 - P_0l_0/E_0 - P_0e_0/E_0$

or

$P_0/E_0 * (1 - l_0 - x_0 + r_0 - l_0r_0 - x_0r_0 - 1 + l_0 + e_0) = d_0 - l_0m_0/E_0 - b_0d_0$

or

$P_0/E_0 = (d_0 - l_0m_0/E_0 - b_0d_0)/(-x_0 + r_0 - l_0r_0 - x_0r_0 + e_0)$

or solving for r_0

$r_0 * (1 - l_0 - x_0) = d_0E_0/P_0 - l_0m_0/P_0 - b_0d_0E_0/P_0 - e_0$

$r_0 = (d_0E_0/P_0 - l_0m_0/P_0 - b_0d_0E_0/P_0 - e_0)/(1 - l_0 - x_0)$,
$r_0 = d_0E_0/P_0$, when l_0, m_0, b_0, e_0, x_0 are zero

The zero-th order equation does not contain the growth rate g, which appears in the first order price, P_1. Therefore, to construct an equation with g, we need to consider the fist order equation. For simplicity, l, x, m, b, and e, which do not appear in the desired equation, are set to zero in the equation for calculating P/E Ratio

$P_0 + P_1 + P_0r_0 + P_1r_0 + P_0r_1 = d_0E_0 + d_0E_1 + d_1E_0 + d_1E_1 + P_0 + P_1(\text{Resale})$

where $P_1(\text{Resale})=P_0*(g_1T + g_2)$. With T=1, the first order equation for the first order rate of return is, neglecting all the smaller terms (P_1, P_1r_0, P_0r_1, d_0E_1, d_1E_0, and d_1E_1),
$r_0 = d_0E_0/P_0 + g_1$
where r_0 is equivalent to equity premium plus riskless interest rate in some derivations. The Infinite Spreadsheet solves the problem exactly.

Summary

The following table summarizes the rigorous derivations in this article and relates the results to other popular rate of return calculations.

Name (Others Names)	Formula	Outcome	Explanation
Zero-th Order (P/E Ratio) (Valuation Multiplier) (Earning-Based) (Yield) (Capitalization Rate)	$r_0 = d_0 * \frac{E_0}{P_0}$ $r_0 = \frac{N_0}{P_0}$	3% (5% Cap Rate)	r_0 = Rate of Return d_0 = Dividend Payout Ratio E_0 = Earning P_0 = Price (Subscript 0 stands for the zero-th order) N_0 = Net Income (r_0 = Cap Rate)
First Order (Expected Rate Of Return) (Gordon Growth Model) (Equity Premium + Riskless Interest Rate) (Dividend-Based)	$r_0 = d_0 * \frac{E_0}{P_0} + g_1$ $r_0 = \frac{N_0}{P_0} + g_1$	73% with 70% Growth (8% with 3% Growth)	g_1 = Growth Rate
Exact Solution (Infinite Spreadsheet) (Cash Flow Model) (Supply Side)	Cash Return = Sum of Cash Flow + Cash From Resale	115% Actual >100%; Price Doubled In the Year (10% Return)	(Price-Loan-BuyerExpense)*[(1+Return) to the power of InvestmentPeriod] = Σ(NetIncome-Loan*%Payment-TaxBracket*NetIncome)+(ResalePrice-Loan-SellerExpense)